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CONFIDENCE LIMITS FOR THE RELIABILITY OF A SERIES SYSTEM

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1. Introduction. We consider a series system which operates if and only if each of k components operates. If the j-th component fails at a random time x according to a probability law given by a cumulative distribution function $F_j(x)$, and if the component failures are statistically independent, then the probability that the system fails before time x is

$$(1.1) \qquad 1 - \prod_{j=1}^{k} (1 - F_j(x)).$$

For the present we will assume that $F_j(x) = 1 - \exp(-x/\theta_j)$, that is, the failure law is exponential with mean time-to-failure equal to θ_j (though this assumption can be relaxed to some degree as indicated in Section 5 below), so that the quantity (1.1) becomes

$$(1.2) \qquad 1 - e^{-\phi x} \qquad \text{where } \phi = \sum_{j=1}^{k} 1/\theta_j .$$

Now suppose that a fixed "mission time" x_o is preassigned and that the system reliability R is defined as the probability of successful operation at least until time x_o. Then

$$(1.3) \qquad R = R(\phi, x_o) = \exp(-\phi x_o) .$$

The present paper is concerned with confidence limits for ϕ, or equivalently for R, based on tests of individual components.

The problem has previously been considered by Rosenblatt (1963), who has given approximate confidence limits based on an assumption of approximately equal sample sizes for the k components. A different approach by Lentner and Buehler (1963) shows how the Lehmann-Scheffé theory of exponential families can be used to find exact confidence limits having certain optimal properties. In that paper, exact solutions are given for the case of two components ($k = 2$), and the solution for general k is indicated. While the analysis is straightforward, the calculations are very heavy for moderate or large sample sizes. The main purpose of the present paper is to give (for arbitrary k) a large-sample approximation to the exact solution (Section 2). It is further shown (Sections 3 and 4) that this approximation agrees with that obtainable from maximum likelihood theory or from a Bayesian analysis. A discussion of the results is given in Section 5.

2. Large-sample approximation to exact confidence limits. Data relating to component mean lifetimes θ_j are presumed to be available in the form of observed gamma variates z_j independently distributed with densities

$$(2.1)\qquad f_{a_j}(z_j;\theta_j) = z^{a_j}\exp(-z_j/\theta_j)/a_j!\theta_j^{a_j+1} \qquad (z_j > 0).$$

As indicated in Section 5 below, z_j might be either the sum of a_j+1 component failure times, or the Epstein-Sobel statistic based on ordered observations. In any case, the joint density of $z_1,\ldots,z_k$ is proportional to

$$(2.2)\qquad \exp\left\{-\sum_{j=1}^{k} z_j/\theta_j\right\}\prod_{j=1}^{k} z_j^{a_j}\ .$$

2.1. The limiting form of the conditional distribution. In order to make inferences about the parametric function

$$(2.3)\qquad \phi = \sum_{j=1}^{k} 1/\theta_j$$

we transform to new variates defined by

$$(2.4)\qquad t = z_1/a_1, \quad r_j = (z_1 - z_j)/a_1 \qquad (j = 2,\ldots,k),$$

obtaining a joint density proportional to

$$(2.5)\qquad \exp\left\{-a_1\phi t + \sum_{j=2}^{k} a_j r_j/\theta_j\right\}\prod_{j=1}^{k}(t - r_j)^{a_j}, \qquad (t \geq r_{max}),$$

where, for notational convenience, we have taken $r_1 = 0$ and $r_{max} = \max(r_1, r_2, \ldots, r_k)$. The conditional density of t given $r_2, \ldots, r_k$ is equal to (2.5) divided by the integral of the same expression over all $t \geq r_{max}$. The factor $\exp(\Sigma a_j r_j/\theta_j)$ is independent of t, and therefore cancels, so that the desired conditional distribution is proportional to

$$(2.6) \qquad \exp(-a_1 \phi t) \prod_{j=1}^{k} (t - r_j)^{a_j}.$$

We wish to consider the case where all of the shape parameters $a_1, \ldots, a_k$ become large simultaneously. For this we define

$$(2.7) \qquad \lambda_j = a_j/a_1 \quad (j = 1, \ldots, k) \quad \text{and} \quad a = a_1,$$

and we let a tend to infinity with $\lambda_2, \ldots, \lambda_k$ fixed. The expression (2.6) can be written in the form

$$(2.8) \qquad \{f(t)\}^a \quad \text{where} \quad f(t) = e^{-\varphi t} \prod_{j=1}^{k} (t - r_j)^{\lambda_j} \quad (t \geq r_{max}),$$

and a theorem of Buehler (1965) enables us to show that the variate t in (2.6) or (2.8) is asymptotically normal as a tends to infinity. To apply the theorem we must first find the mode (i.e., maximum) of $f(t)$, show $f(t)$ is unimodal, and find the second derivative at the mode.

It is seen from (2.8) that f tends to zero at both limits of its domain (r_{max}, ∞), so that it must have a maximum at some point $t = m$ inside the domain for which $f'(m) = 0$, or equivalently,

$$(2.9) \qquad \sum_{j=1}^{k} \lambda_j/(m - r_j) = \phi.$$

We will now show that only one value of m in (r_{max}, ∞) satisfies (2.9). To this end let m be any solution of (2.9) and put $x = t-m$. Then

$$(2.10) \qquad \frac{f(t)}{f(m)} = \frac{f(x+m)}{f(m)} = e^{-\phi x} \prod \left\{ 1 + \frac{x}{m-r_j} \right\}^{\lambda_j} .$$

Using the inequality $1+y \leq e^y$ k times with $y = x/(m-r_j)$ gives (in the range $t > r_{max}$ where all factors are positive)

$$(2.11) \qquad \frac{f(x+m)}{f(m)} \leq e^{-\phi x} \prod \exp\left\{ \frac{\lambda_j x}{m-r_j} \right\} = \exp\left\{ \left[\Sigma \frac{\lambda_j}{m-r_j} - \phi \right] x \right\} = 1$$

where at the last step we have used (2.9). Equality holds only if $x = 0$. If (2.9) has two distinct solutions m_1 and m_2, the above argument applied to each gives the contradictory $f(m_2) < f(m_1)$ and $f(m_1) < f(m_2)$, so that (2.9) is satisfied by a unique m, and f is unimodal.

Now let us put $\sigma^{-2} = f''(m)/f(m) = g''(m)$ where $g(t) = \log f(t)$. Then

$$(2.12) \qquad \sigma^{-2} = \sum_{j=1}^{k} \lambda_j/(m-r_j)^2 .$$

If t is a variate whose density is proportional to (2.6) or (2.8), if m is given implicitly as the unique root of (2.9) in the range (r_{max},∞), then it follows from Buehler (1965) that the variate $a^{\frac{1}{2}}(t - m)/\sigma$ tends in distribution to standard normal as a tends to infinity. To express this result by an equation, let us first define the normal percentile point ξ_γ by

$$(2.13) \qquad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\xi_\gamma} e^{-t^2/2} dt = \gamma .$$

Then as $a \to \infty$,

$$(2.14) \qquad \lim P\{a^{\frac{1}{2}}(t - m)/\sigma \leq \xi_\gamma | r_2,\ldots,r_k;\theta_1,\ldots,\theta_k\} = \gamma .$$

2.2 Approximate confidence limits. Let $F(t|r_2,\ldots,r_k,\lambda_2,\ldots,\lambda_k;\phi)$ denote the cumulative distribution function corresponding to the density proportional to (2.6) or (2.8). For any confidence level γ, exact upper confidence limits for ϕ are obtained by solving for ϕ the implicit equation $F(t|r_2,\ldots,r_k,\lambda_2,\ldots,\lambda_k;\phi) = \gamma$ to give $\phi = \bar{\phi}(t,r_2,\ldots,r_k,\lambda_2,\ldots,\lambda_k;\gamma)$. This is so since $P(\phi \leq \bar{\phi}) = P(F \leq \gamma) = \gamma$, conditionally, given $r_2,\ldots,r_k$, and hence also unconditionally. Since this exact solution involves very heavy numerical calculations, we consider an approximate solution based on (2.14).

Let us define $\bar{\phi}_1 = \bar{\phi}_1(t,r_2,\ldots,r_k a,\lambda_2,\ldots,\lambda_k;\gamma)$ to be the value of ϕ obtained from

$$(2.15) \qquad a^{\frac{1}{2}}(t - m(\phi))/\sigma(\phi) = \xi_\gamma$$

where $m(\phi)$ and $\sigma(\phi)$ are the simultaneous solutions of (2.9) and (2.12) (depending also on $r_2,\ldots,r_k,\lambda_2,\ldots,\lambda_k$, which are supressed in the notation). In other words, $\bar{\phi}_1$ is obtained by eliminating m and σ from equations (2.9), (2.12) and (2.15). Since the inequality $\phi \leq \bar{\phi}_1$ is equivalent to $a^{\frac{1}{2}}(t - m)/\sigma \leq \xi_\gamma$, we have from (2.14),

$$(2.16) \qquad \lim P\{\phi \leq \bar{\phi}_1|r_2,\ldots,r_k;\theta_1,\ldots,\theta_k\} = \gamma \quad \text{as} \quad a \to \infty.$$

Thus $\bar{\phi}_1$ gives approximate confidence limits in the sense that the probability of covering the true value approaches the stated confidence level γ as $a \to \infty$, for all $\theta_1,\ldots,\theta_k$, given fixed $r_2,\ldots,r_k$. It is of interest to know whether the same result holds for the unconditional probability in which $r_2,\ldots,r_k$ are not fixed. An affirmative answer follows from the easily verifiable fact that r_j tends in probability to the constant $\theta_1 - \lambda_j\theta_j$ as $a \to \infty$ and from the Lemma given in the Appendix.

Although it is not possible to eliminate m and σ from (2.9), (2.12) and (2.15) to give an explicit formula for $\phi = \bar{\phi}_1$, it is possible to obtain expressions as power series in $a^{-\frac{1}{2}}$. From (2.15),

$m = t - \sigma\xi_\gamma a^{-\frac{1}{2}}$, from which

$$(2.17) \qquad (m-r)^{-1} = (t-r)^{-1}\{1 - \sigma\xi_\gamma a^{-\frac{1}{2}}(t-r)^{-1}\}^{-1}$$

$$= (t-r)^{-1} + (t-r)^{-2}\sigma\xi_\gamma a^{-\frac{1}{2}} + O(a^{-1})$$

and substituting in (2.9) gives

$$(2.18) \qquad \emptyset = \Sigma \lambda_j(m - r_j)^{-1}$$

$$= \Sigma \lambda_j(t - r_j)^{-1} + a^{-\frac{1}{2}}\sigma\xi_\gamma \Sigma \lambda_j(t - r_j)^{-2} + O(a^{-1}).$$

From (2.12) we have

$$(2.19) \qquad \begin{aligned} \sigma &= \{\Sigma \lambda_j(m - r_j)^{-2}\}^{-\frac{1}{2}} \\ &= \{\Sigma \lambda_j(t - r_j - a^{-\frac{1}{2}}\sigma\xi_\gamma)^{-2}\}^{-\frac{1}{2}} \\ &= \{\Sigma \lambda_j(t - r_j)^{-2} + O(a^{-\frac{1}{2}})\}^{-\frac{1}{2}} \\ &= \{\Sigma \lambda_j(t - r_j)^{-2}\}^{-\frac{1}{2}} + O(a^{-\frac{1}{2}}). \end{aligned}$$

Combining (2.18) and (2.19),

$$(2.20) \qquad \emptyset = \overline{\emptyset}_1 = \Sigma \lambda_j(t - r_j)^{-1} + a^{-\frac{1}{2}}\xi_\gamma \{\Sigma \lambda_j(t - r_j)^{-2}\}^{\frac{1}{2}} + O(a^{-1}).$$

Returning via (2.4) and (2.7) to the original variables gives

$$(2.21) \qquad \overline{\emptyset}_1 = \Sigma a_j/z_j + \xi_\gamma\{\Sigma a_j/z_j^2\}^{\frac{1}{2}} + O(a^{-1}).$$

Finally, we may substitute $\overline{\emptyset}_1$ for $\emptyset$ in (1.3) to obtain a confidence limit for the reliability R. Discarding terms of $O(a^{-1})$ we obtain a lower confidence limit corresponding to confidence level γ of the form

$$(2.22) \qquad \underline{R} = \exp\{- x_o \Sigma a_j/z_j\}\{1 - \xi_\gamma x_o[\Sigma a_j/z_j^2]^{\frac{1}{2}}\}.$$

3. Maximum likelihood theory. If the joint distribution of statistics $T_1,\ldots,T_k$ tend to k-variate normal with means $\theta_1,\ldots,\theta_k$ and dispersion matrix $n^{-1}(\sigma_{ij})$ where the σ_{ij} do not depend on n, and if $\beta(T_1,\ldots,T_k)$ is a continuous function with continuous partial derivatives, then it is known (see for example Rao (1952)) that $\beta(T_1,\ldots,T_k)$ is asymptotically normal with mean $\beta(\theta_1,\ldots,\theta_k)$ and variance $n^{-1}\Sigma\Sigma\sigma_{ij}(\partial\beta/\partial\theta_i)(\partial\beta/\partial\theta_j)$. In the present case we take T_j to be the maximum likelihood estimator of θ_j in (2.1),

$$(3.1) \qquad T_j = \hat{\theta}_j = z_j/n_j \,, \qquad (n_j = a_j + 1),$$

for which we easily find $ET_j = \theta_j$, $\text{Var}\ T_j = \theta_j^2/n_j$, $\text{Cov}(T_i,T_j) = 0$ $(i \neq j)$. If we put

$$(3.2) \qquad \beta(\theta_1,\ldots,\theta_k) = \phi = \Sigma\ 1/\theta_j \,,$$

and if we suppose that $n_j = \rho_j n_1$ where the ρ_j are fixed as $n_1 \to \infty$, then the asymptotic variance of $\hat{\phi} = \beta(T_1,\ldots,T_k)$ equals

$$(3.3) \qquad n^{-1}\ \Sigma\ \sigma_{ii}(\partial\beta/\partial\theta_i)^2 = n^{-1}\ \Sigma\ (\theta_j^2/\rho_j)(-1/\theta_j^2)^2 = \Sigma\ 1/(n_j\theta_j^2), \quad n = n_1,$$

and the estimated variance is $\hat{\sigma}^2 = \Sigma\ 1/(n_j\hat{\theta}_j^2) = \Sigma\ n_j/z_j^2$. By these approximations, an upper confidence limit for ϕ with confidence level γ is

$$(3.4) \qquad \bar{\phi}_2 = \phi + \xi_\gamma\hat{\sigma} = \Sigma\ n_j/z_j + \xi_\gamma\{\Sigma\ n_j/z_j^2\}^{\frac{1}{2}} .$$

Comparison with (2.21) shows that the right hand side of (3.4) corresponds with the first two terms in the right hand side of (2.21) with n_j replacing a_j (where $n_j = a_j+1$). To make comparisons for large n_j, the "stochastically large" z_j should be replaced by substituting $z_j = n_j r_j$. When this is done it is seen that $\bar{\phi}_1 - \bar{\phi}_2 = O(n^{-1})$, so that the two solutions agree to order n^{-1}.

4. <u>A Bayesian solution</u>. For Bayesian estimation of the scale parameter θ of (2.1), we arbitrarily take the usual "improper" prior density $p(\theta)d\theta = d\theta/\theta$, $0 < \theta < \infty$. It is convenient to define $\alpha = 1/\theta$, and it is easily shown that the posterior distribution of α given z is a gamma distribution with $E\alpha = n/z$ and $\text{Var}\ \alpha = n/z^2$. If θ_j $(j = 1,\dots,k)$ have independent prior densities $d\theta_j/\theta_j$, then the posterior distribution of $\phi = \Sigma(1/\theta_j) = \Sigma\ \alpha_j$ is the sum of independent gamma variates. It is known that each of these is asymptotically normal as n_j tends to infinity, and it follows that if all the n_j increase at the same rate, the sum is asymptotically normal. Since the posterior mean and variance are seen to be $E\phi = \Sigma E\alpha_j = \Sigma(n_j/z_j)$ and $\text{Var}\ \phi = \Sigma\text{Var}\ \alpha_j = \Sigma(n_j/z_j^2)$, an approximate Bayesian upper confidence limit with confidence level γ is

$$\bar{\phi}_3 = \Sigma(n_j/z_j) + \xi_\gamma\{\Sigma(n_j/z_j^2)\}^{\frac{1}{2}}, \tag{4.1}$$

which is identical with $\bar{\phi}_2$ given by (3.4).

5. <u>Discussion</u>. The present paper is concerned with one aspect of the general problem of inference in which conclusions are desired about the reliability of a system of several components and data are available which furnish information about the reliability of the individual components. For a good review of earlier work in this general area, the reader is referred to Rosenblatt (1963).

In the analysis given above we have restricted our attention to the case of a series system of k dissimilar components, and have considered the problem of obtaining confidence limits for the "reliability" of the system, where "reliability" is defined as the probability of successful operation at least until a given preassigned mission time x_o. If we suppose for the moment that component j $(j = 1,\dots,k)$ follows an exponential failure law with mean life θ_j, that n_j units of type j have previously been tested, and their total liketimes have been z_j, then it readily follows from the reproductive property of the gamma distribution that z_j has the density (2.1) with $a_j = n_j-1$, Alternatively, if only the first r_j failures are observed in tests of n_j items, then it is known that the statistic

$$(5.1) \qquad z_j = \sum_{i=1}^{r_j} x_{ji} + (n_j - r_j)x_{jr_j}$$

(where $x_{j1},\ldots, x_{jr_j}$ denote ordered observations) has the density (2.1) with r_j substituted for n_j (Epstein and Sobel (1953)). Using the statistics $z_1,\ldots,z_k$ it is possible to determine exact confidence limits for the mean life ϕ^{-1} (where $\phi = \Sigma\ 1/\theta_i$) of the k-component system, or for the reliability $R = \exp(-\phi x_o)$. The exact solution utilizes the Lehmann-Scheffé theory of exponential families, is analogous to that given by Lentner and Buehler (1963) for $k = 2$, and the results can be found in El Mawaziny (1965). In addition to their exactness, these solutions are known to have the desirable property of being "uniformly most accurate unbiased" confidence limits as defined by Lehmann (1959), Section 5.5. Because the exact solution involves rather heavy calculations, both algebraic and numerical, the present paper is devoted to approximations valid for the case of large samples. The results are derived in Section 2, with the approximate confidence limits for ϕ and for R being given in equations (2.21) and (2.22). In Sections 3 and 4 it is shown that the above approximations agree with those that are obtainable from maximum likelihood theory or from a Bayesian solution.

Thus far we have supposed that the individual components have exponential time-to-failure. Within limits, it is possible to relax this assumption by the method of variate transformation. Thus if it is known that time-to-failure y is so distributed that a known function, say g(y), is exponentially distributed, then the analysis may proceed with $x = g(y)$. This is the case with the Weibull distribution, for example, with known "shape" parameter. However, as Lentner and Buehler (1963) have indicated, in order for the exact confidence limit theory to be applicable, it is necessary that the same function $g(y)$ must apply to each of the components.

What can be said about systems other than series systems--for example, parallel systems? So far as we are aware, exact solutions of the type we have been discussing for the series system are not possible in other cases, and the analysis of Section 2 does not extend to more general systems. On the other hand, approximate solutions of various kinds would certainly be possible. For example, maximum likelihood theory could be employed as in Section 3 above, in quite general circumstances, to give large-sample approximations.

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APPENDIX

UNCONDITIONAL LIMITS OF CONDITIONAL PROBABILITIES

In this appendix we show that if confidence limits are calculated from conditional distributions in such a way that the conditional probability of coverage of the true value approaches the desired confidence level, then so does the unconditional probability, provided the conditioning variates tend in probability to constants. The result is used in Section 2.

<u>Lemma</u>. Let $(x_1,\ldots,x_m,y)$ be coordinates in Euclidean space R_{m+1}. Assume that: (i) $\{P_n\}$ is a sequence of probability measures on R_{m+1} representable as

$$(A.1) \qquad dP_n = f_n(\underset{\sim}{x})g_n(y|\underset{\sim}{x})\, d\underset{\sim}{x}\, dy, \qquad \underset{\sim}{x} = (x_1,\ldots,x_m),$$

where $g_n(y|\underset{\sim}{x})$ is a conditional density; (ii) the measures $\{P_n\}$ have the property that for some finite constants c_j,

$$(A.2) \qquad x_j \to c_j \text{ in probability}, \quad j = 1,\ldots,m, \quad \text{as } n \to \infty;$$

(iii) $\{S_n\}$ is a sequence of subsets of R_{n+1} such that

$$(A.3) \qquad P_n\{S_n|\underset{\sim}{x}\} \to \gamma \quad \text{for each } x \text{ as } n \to \infty,$$

where $P_n(\cdot|\underset{\sim}{x})$ denotes conditional probability with respect to the density g_n. Then

$$(A.4) \qquad P_n\{S_n\} \to \gamma \quad \text{as } n \to \infty.$$

<u>Proof</u>: Define cylinder sets in R_m by

$$(A.5) \qquad B_j = \{x_1,\ldots,x_m \big| \, |x_j - c_j| \le 1\} \qquad j = 1,\ldots,m,$$

and put $B = \bigcap B_j$, $B' = \bigcup B_j'$, where "prime" denotes complement.

For any $\epsilon > 0$, by (ii) we can find $N = N(\epsilon)$ such that $P_n\{B_j'\} \leq \epsilon/4m$ for $n > N(\epsilon)$ and $j = 1,\ldots,m$. Thus $P_n\{B'\} = P_n\{\bigcup B_j'\} \leq \Sigma P_n\{B_j\} \leq \epsilon/4$ for $n > N$. Since B is a compact subset of R_m we can by (iii) find $N' = N'(\epsilon)$ such that $|P_n\{S_n|\underset{\sim}{x}\} - \gamma| \leq \frac{\epsilon}{2}$ for all $x \in B$ and all $n > N'$.
Thus

$$|P_n\{S_n\} - \gamma| = |\int P_n\{S_n|\underset{\sim}{x}\} f_n(\underset{\sim}{x})d\underset{\sim}{x} - \gamma| \tag{A.6}$$

$$\leq \int |P_n\{S_n|\underset{\sim}{x}\} - \gamma|\ f_n(\underset{\sim}{x})d\underset{\sim}{x}.$$

The last expression is the sum of

$$\int_B |\ P_n\{S_n|\underset{\sim}{x}\} - \gamma|\ f_n(\underset{\sim}{x})d\underset{\sim}{x} \leq \frac{\epsilon}{2} \int_B f_n(\underset{\sim}{x})d\underset{\sim}{x} \leq \frac{\epsilon}{2} \quad \text{for } n > N' \tag{A.7}$$

and

$$\int_{B'} |P_n\{S_n|\underset{\sim}{x}\} - \gamma|\ f_n(\underset{\sim}{x})d\underset{\sim}{x} \leq 2 \int_{B'} f_n(\underset{\sim}{x})d\underset{\sim}{x} = 2P_n\{B'\} \leq \frac{\epsilon}{2} \quad \text{for } n > N. \tag{A.8}$$

Thus for $n > \max(N,N')$, $|P_n\{S_n\} - \gamma| < \epsilon$.